WEIGHTED NORM INEQUALITIES FOR COMPOSITION OF OPERATORS ASSOCIATED WITH DIFFERENT HOMOGENEITIES

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Abstract. We study the questions of the composition of operators which can not be answered by using the properties of each operator separately. Phong and Stein studied the week type $(1,1)$ estimate while Han et al. considered the Hardy space for the composition of operators with different homogeneities. In this paper, we characterize a class of the Muckenhoupt $A^p_\varphi$ weights associated with different homogeneities and prove the $H^p_{\varphi,w}$ weighted norm inequalities for composition of operators which cover the operators studied in \cite{PS} and \cite{HLLRS}.

1. Introduction and statement of main results

Phong and Stein in \cite{PS} studied the compositions of singular integral operators with different homogeneities arising naturally in $\bar{\partial}$-Neumann problem. Recently, the Hardy spaces associated with different homogeneities were introduced in \cite{HLLRS}. The purpose of this paper is to characterize a class of the Muckenhoupt weights $A^p_\varphi$ and prove the weighted norm inequalities for the composition of operators associated with different homogeneities on $\mathbb{R}^N$. When $w \in A^\infty_\varphi$, we establish the weighted $H^p_{\varphi,w} - H^p_{\varphi,w}$ and $H^p_{\varphi,w} - L^p_w$ boundedness of the compositions of singular integrals with “minimal” regular and cancellation conditions, which cover the operators studied in \cite{PS} and \cite{HLLRS}.

The composition of operators was considered by Calderón and Zygmund when they introduced the first generation of Calderón-Zygmund convolution operators. Indeed, to compose two convolution operators, $T_1$ and $T_2$, it is enough to employ the product of the corresponding multipliers $m_1(\xi)$ and $m_2(\xi)$. However, the symbol $m_3(\xi) = m_1(\xi)m_2(\xi)$ does not necessarily have zero integral on the unit sphere, so Calderón and Zygmund considered the algebra of operators $cI + T$, where $c$ is a constant, $I$ is the identity operator and $T$ is the operator introduced by them. In 1965, Calderón considered again the problem of the symbolic calculus of the second generation of Calderón-Zygmund singular integral

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operators with the minimal regularity with respect to \( x \) on kernels \( L_1(x, y) \) and \( L_2(x, y) \), corresponding to operators \( T_1 \) and \( T_2 \). This problem led to the study of the commutator.

In the present paper, we consider the composition of two operators associated with different homogeneities. To be more precise, let \( e(\xi) \) be a function on \( \mathbb{R}^n \) homogeneous of degree 0 in the isotropic sense and smooth away from the origin. Similarly, suppose that \( h(\xi) \) is a function on \( \mathbb{R}^n \) homogeneous of degree 0 in the non-isotropic sense related to the heat equation, and also smooth away from the origin. Then it is well-known that the Fourier multipliers \( T_1 \) defined by \( \hat{T}_1(f)(\xi) = e(\xi) \hat{f}(\xi) \) and \( T_2 \) given by \( \hat{T}_2(f)(\xi) = h(\xi) \hat{f}(\xi) \) are both bounded on \( L^p \) for \( 1 < p < \infty \), and satisfy various other regularity properties such as being of weak-type \((1, 1)\) and bounded on the classical isotropic and non-isotropic Hardy spaces, respectively. Rivière in [WW] asked the question: is the composition \( T_1 \circ T_2 \) still of weak-type \((1, 1)\)? Phong and Stein in [PS] answered this question and gave a necessary and sufficient condition for which \( T_1 \circ T_2 \) is of weak-type \((1, 1)\). The operators Phong and Stein studied are in fact compositions with different kind of homogeneities which arise naturally in the \( \bar{\partial} \)-Neumann problem. Recently, Han etc. [HLLRS] have developed a theory of Hardy spaces and proved that the composition of two Calderón-Zygmund singular integral operators with different homogeneities is bounded on these new Hardy spaces. The inhomogeneous Besov spaces and Triebel-Lizorkin spaces associated with different homogeneities were established in [WL].

In this paper, we study the weighted norm inequalities for operators associated with different homogeneities. We first introduce a class of Muckenhoupt weights \( A_p^\varphi \) associated with different homogeneities and give characterizations of these weights (see Theorem 1.2 below). In terms of these \( A_p^\varphi \) weights, we develop the theory of the weighted Hardy space \( H_{p,w}^\varphi(\mathbb{R}^N) \) via Littlewood-Paley theory and prove the \( H_{p,w}^\varphi(\mathbb{R}^N) \to H_{p,w}^\varphi(\mathbb{R}^N) \) and \( H_{p,w}^\varphi(\mathbb{R}^N) \to L_p^w(\mathbb{R}^N) \) boundedness of composition operators with different homogeneities. We point out that the whole theory of weighted Hardy spaces are closely related to the geometric structure of \( \mathbb{R}^N \) with mixed homogeneities, namely, a family of rectangles with acceptable size, will play a significant role.

In order to describe more precisely questions and results studied in this paper, we begin with considering all functions and operators on \( \mathbb{R}^N = \mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_m} \). For \( x = (x_1, \ldots, x_m) \in \mathbb{R}^N \) and \( \delta > 0 \), we consider two kinds of homogeneities on \( \mathbb{R}^N \)

\[
\delta \circ_1 (x_1, \ldots, x_m) = (\delta^{a_1} x_1, \ldots, \delta^{a_m} x_m),
\]

and

\[
\delta \circ_2 (x_1, \ldots, x_m) = (\delta^{b_1} x_1, \ldots, \delta^{b_m} x_m),
\]
with $0 < a_1 \leq \ldots \leq a_m < \infty$ and $0 < b_1 \leq \ldots \leq b_m < \infty$. Let $N_1 = a_1 n_1 + \ldots + a_m n_m$ and $N_2 = b_1 n_1 + \ldots + b_m n_m$ denote the homogeneous dimensions of $\mathbb{R}^N$ with respect to $\circ_1$ and $\circ_2$, respectively. For $x = (x_1, \ldots, x_m) \in \mathbb{R}^N$, we denote $|x|_1 = (|x_1|^{\frac{2}{n_1}} + \ldots + |x_n|^{\frac{2}{n_n}})^{\frac{1}{2}}$ and $|x|_2 = (|x_1|^{\frac{2}{n_1}} + \ldots + |x_n|^{\frac{2}{n_n}})^{\frac{1}{2}}$ be homogeneous norms with respect to these non-isotropic dilations, respectively. We also use the discrete dilations: for $j, k \in \mathbb{Z}$,
\[ 2^j \circ_1 (x_1, \ldots, x_m) \to (2^j x_1, \ldots, 2^j x_m), \]
and
\[ 2^k \circ_2 (x_1, \ldots, x_m) \to (2^k x_1, \ldots, 2^k x_m). \]

For $j \in \mathbb{Z}$ and $i = 1, \ldots, m$, let $j_i = a_i j$ and $j = (j_1, \ldots, j_m)$. Similarly, for $k \in \mathbb{Z}$, let $k_i = b_i k$ and $k = (k_1, \ldots, k_m)$. Let $2^{j \vee k} = (2^{j_1 \vee k_1}, \ldots, 2^{j_m \vee k_m})$. We also use notations $j \wedge k = \min\{j, k\}$ and $j \lor k = \max\{j, k\}$. Denote by $I_1$ and $I_2$ the sets of all cubes in $\mathbb{R}^n$ with side length $2^j$ and $2^k$, respectively.

It is well known that the Muckenhoupt class of $A_p$ weight appears when one tries to determine the weight functions $w$ for which the Hardy-Littlewood maximal function (associated with one-parameter dilations) to be bounded on the weighted Lebesgue space $L^p_w$. For non-isotropic dilation $\circ_1$, the (centered) Hardy-Littlewood maximal function $M_{(1)}$ is defined by
\[ M_{(1)} f(x) = \sup_{I \in I_1} \frac{1}{|I|} \int_I |f(y)| dy, \]
where $I_1$ runs over all “cubes” centered at $x$ with side length $2^j, \ldots, 2^m$. It is well known that the maximal function $M_{(1)}$ is bounded on $L^p_w, 1 < p < \infty$ if and only if $w$ satisfies the following $A_p^{(1)}$ condition:
\[ \left( \int_I w(x) dx \right) \left( \int_I w(x)^{-1/(p-1)} dx \right)^{p-1} < \infty. \]
Similar definitions and results hold for $M_{(2)}$ and $A_p^{(2)}$ (with respect to $\circ_2$).

Another important operators associated with these dilations are singular integral operators defined by

**Definition 1.1.** Let $i = 1, 2$. A locally integrable function $K_i$ on $\mathbb{R}^N \setminus \{0\}$ is said to be a Calderón-Zygmund kernel associated with $\circ_i$ if it satisfies
\begin{align*}
(1.1) & \quad |K_i(x)| \leq A|x_i|^{-N_i}, \quad \text{for } x \in \mathbb{R}^N \setminus \{0\}; \\
(1.2) & \quad |K_i(x + u) - K_i(x)| \leq A|u_i|^{\varepsilon_i}/x_i^{-N_i + \varepsilon_i}, \quad \text{if } |x_i| \geq 2|u_i|; \\
(1.3) & \quad \int_{|s| < |x_i| < r} K_i(x) dx \leq C, \quad \text{uniformly for all } r, \delta > 0.
\end{align*}
We say that an operator $T_i$ is a singular integral operator associated with the $\circ_i$ homogeneity if $T_i(f)(x) = p.v. (\mathcal{K}_i * f)(x)$, where $\mathcal{K}_i$ satisfies conditions of (1.1)-(1.3).

In the current setting for compositions, it is natural to consider the following question: for what kind of weight function $w$, the composition of two Hardy-Littlewood maximal function $\mathcal{M}(1) \circ \mathcal{M}(2)$ or the composition of two Calderón-Zygmund operators $T_1 \circ T_2$ are bounded on $L^p_w$?

If $w \in A_p^{(1)} \cap A_p^{(2)}(\mathbb{R}^N)$, then both $\mathcal{M}(1) \circ \mathcal{M}(2)$ and $T_1 \circ T_2$ are bounded on $L^p_w$. Conversely, if for all Calderón-Zygmund operators $T_1$ and $T_2$ associated to dilations $\circ_1$ and $\circ_2$, respectively, $T_1 \circ T_2$ are bounded on $L^p_w$, then we can take $T_1 = I$ or $T_2 = I$, the identity operator on $\mathbb{R}^N$, then we see that $w$ must be in $A_p^{(1)} \cap A_p^{(2)}$; We shall show that this is also true for $\mathcal{M}(1) \circ \mathcal{M}(2)$. Moreover, the weights can be characterized by a new maximal function and relative weights associated with a family of rectangles of acceptable size, which reflect the geometry structure of $\mathbb{R}^N$ with different homogeneities.

We say that a rectangle $R$ is of acceptable size (or a $C$-rectangle), if $R = I_1 \times \cdots \times I_m$, where $I_i$'s are usual cubes in $\mathbb{R}^n$ with side-length $2^{j_i \vee k_i} = 2^{(a_i, j) \vee (b_i, k)}$, $j, k \in \mathbb{Z}$. Denote by $\mathcal{R}_C$ the set of all $C$-rectangles. For $j, k \in \mathbb{Z}$, let $\mathcal{R}_{C}^{j,k}$ be the subset of $\mathcal{R}_C$ that consists of all $C$-rectangle with side-length $(2^{j_1 \vee k_1}, 2^{j_2 \vee k_2}, \ldots, 2^{j_m \vee k_m})$.

We next introduce the Muckenhoupt class of $A_p$ weights associated with different homogeneities.

**Definition 1.2.** If $1 < p < \infty$, let $w$ be a nonnegative locally integrable function on $\mathbb{R}^N$. We say that $w$ is in $A_p^\varphi(\mathbb{R}^N)$ if there is a constant $C$ such that

$$\sup_{R \in \mathcal{R}_C} \left( \frac{1}{|R|} \int_R w(x) \, dx \right) \left( \frac{1}{|R|} \int_R w(x)^{-1/(p-1)} \, dx \right)^{p-1} < \infty.$$ 

The weight class $A_p^\infty(\mathbb{R}^N)$ is defined by $A_p^\infty(\mathbb{R}^N) = \bigcup_{1 < p < \infty} A_p^\varphi(\mathbb{R}^N)$. We use $q_w := \inf \{ q : w \in A_q^\varphi(\mathbb{R}^N) \}$ to denote the critical index of $w$.

**Definition 1.3.** The maximal function associated to the $C$-rectangles is defined by

$$\mathcal{M}_C(f)(x) = \sup \frac{1}{|R|} \int_R |f(y)| \, dy,$$

where the supremum are taken over all $R \in \mathcal{R}_C$ centered at $x$.

**Remark 1.1.** The Muckenhoupt class of $A_1$ weights can be defined via maximal function as follows. A nonnegative locally integrable function $w$ is in $A_1^\varphi(\mathbb{R}^N)$ if there is a constant $C$ such that $\mathcal{M}_C(w)(x) \leq Cw(x)$ for almost every $x \in \mathbb{R}^N$. 
Now let us state our first main result in this paper.

**Theorem 1.1.** Let $1 < p < \infty$. The following four statements are equivalent:

1. $w \in A_p^C(\mathbb{R}^N)$;
2. $w \in A_p^{(1)}(\mathbb{R}^N) \cap A_p^{(2)}(\mathbb{R}^N)$;
3. $M_{(1)} \circ M_{(2)}$ is bounded on $L_p^w(\mathbb{R}^N)$;
4. $M_c$ is bounded on $L_p^w(\mathbb{R}^N)$.

From Theorem 1.1, one can easily verify that most basic properties for classical $A_p$ weight with respect to cubes, such as reverse Hölder’s inequalities and doubling properties, also hold for $A_p^C$ weight with respect to $C$-rectangles. We refer to the reader [GR, S] for more details about the theory of Muckenhoupt weights.

From Theorem 1.1 and its proof, we also have the following Fefferman-Stein vector-valued inequality, which will be a key tool in proving the main theorems below.

**Corollary 1.1.** The following Fefferman-Stein vector-valued inequality holds

$$
\int_{\mathbb{R}^N} |M_c(f)(x)|_p^p w(x) dx \leq C \int_{\mathbb{R}^N} |f(x)|_p^p w(x) dx, \quad 1 < p, q < \infty.
$$

if and only if $w \in A_p^C(\mathbb{R}^N)$, provided that $f = (f_1, f_2, \cdots) \in L_p^w(\ell^q)$.

When $w \in A_p^{(1)}(\mathbb{R}^N)$, the composition $T_1 \circ T_2$ is bounded on $L_p^w, 1 < p < \infty$ since each $T_i$ is. However, for $0 < p \leq 1$, weighted norm inequality for Hardy spaces can not be established by using the boundedness properties of each operator separately (see Remark 1.2 (i) below). Our next target is to develop a theory of weighted Hardy spaces associated with different homogeneities and prove the boundedness of composition $T_1 \circ T_2$ on these new weighted Hardy spaces.

Let $\psi^{(1)} \in S(\mathbb{R}^N)$ satisfy

$$(1.4) \quad \text{supp} \hat{\psi}^{(1)}(\xi) \subset \{ \xi : 1/2 < |\xi| \leq 2 \},$$

and

$$(1.5) \quad \sum_{j \in \mathbb{Z}} \hat{\psi}^{(1)}(2^j \circ \xi) = 1, \quad \text{for all } \xi \in \mathbb{R}^N \setminus \{0\}.$$  

and let $\psi^{(2)} \in S(\mathbb{R}^N)$ satisfy

$$(1.6) \quad \text{supp} \hat{\psi}^{(2)}(\xi) \subset \{ \xi : 1/2 < |\xi| \leq 2 \},$$

and

$$(1.7) \quad \sum_{k \in \mathbb{Z}} \hat{\psi}^{(2)}(2^k \circ \xi) = 1, \quad \text{for all } \xi \in \mathbb{R}^N \setminus \{0\}.$$
For \( f \in S'/P(\mathbb{R}^N) \), the Littlewood-Paley-Stein square function \( g_\varepsilon(f) \) of \( f \) is defined by

\[
g_\varepsilon(f)(x) = \left\{ \sum_{j,k \in \mathbb{Z}} \sum_{R \in \mathcal{R}_j^k} |\psi_{j,k} * f(x_R)|^2 \chi_R(x) \right\}^{\frac{1}{2}},
\]

where \( \psi_j^{(1)}(x) = 2^{-jN_1}\psi(2^{-j-1}x) \), \( \psi_k^{(2)}(x) = 2^{-kN_2}\psi(2^{-k-1}x) \) and \( \psi_{j,k} = \psi_j^{(1)} * \psi_k^{(2)} \).

The weighted Hardy space associated with different homogeneities is given by the following

**Definition 1.4.** Let \( 0 < p < \infty \) and \( w \in A^p_\infty(\mathbb{R}^N) \), the weighted Hardy space \( H^p_{\varepsilon,w}(\mathbb{R}^N) \) is defined by

\[
H^p_{\varepsilon,w}(\mathbb{R}^N) = \{ f \in S'/P(\mathbb{R}^N) : g_\varepsilon(f) \in L^p_w(\mathbb{R}^N) \}.
\]

The \( H^p_{\varepsilon,w}(\mathbb{R}^N) \) (quasi-)norm of \( f \) is given by \( \|f\|_{H^p_{\varepsilon,w}(\mathbb{R}^N)} \equiv \|g_\varepsilon(f)\|_{L^p_w(\mathbb{R}^N)}. \)

To see the definition of weighted Hardy space \( H^p_{\varepsilon,w} \) is independent of \( \psi_{j,k} \), we prove the following

**Theorem 1.2.** Let \( 0 < p < \infty \) and \( w \in A^p_\infty(\mathbb{R}^N) \). Suppose \( \psi_{j,k}, \varphi_{j,k} \) satisfy the conditions \([1.4] - [1.7] \). Then we have

\[
\left\| \sum_{j,k \in \mathbb{Z}} \sum_{R \in \mathcal{R}_j^k} |\psi_{j,k} * f(x_R)|^2 \chi_R(\cdot) \right\|_{L^p_w(\mathbb{R}^N)} \approx \left\| \sum_{j,k \in \mathbb{Z}} \sum_{R \in \mathcal{R}_j^k} |\varphi_{j,k} * f(x_R)|^2 \chi_R(\cdot) \right\|_{L^p_w(\mathbb{R}^N)},
\]

where each \( x_R \) denotes the “minimal corner” of \( R \), i.e., the corner of \( R \) which has the least value of each coordinate component.

The second main result in this paper is the following

**Theorem 1.3.** Let \( w \in A^p_\infty(\mathbb{R}^N) \). Then for \( p > q \), \( 1 \leq q \leq p \), and \( \varepsilon \), \( \varepsilon' \) satisfying \([1.4] - [1.7] \), the composition operator \( T_1 \circ T_2 \) is bounded on \( H^p_{\varepsilon,w}(\mathbb{R}^N) \). Moreover, there exists a constant \( C \) depending only on the dimensions and \( w \) such that

\[
\| (T_1 \circ T_2)f \|_{H^p_{\varepsilon,w}(\mathbb{R}^N)} \leq C \|f\|_{H^p_{\varepsilon,w}(\mathbb{R}^N)}.
\]

**Remark 1.2.** (i) There are two Littlewood-Paley-Stein square function \( g_1 \) and \( g_2 \) with respect to \( \circ_1 \) and \( \circ_2 \), respectively, defined by

\[
g_1(f)(x) = \left\{ \sum_{j \in \mathbb{Z}} |\psi_j^{(1)} * f(x)|^2 \right\}^{\frac{1}{2}} \quad \text{and} \quad g_2(f)(x) = \left\{ \sum_{k \in \mathbb{Z}} |\psi_k^{(2)} * f(x)|^2 \right\}^{\frac{1}{2}},
\]

where \( \psi_j^{(1)} \) and \( \psi_k^{(2)} \) are the same functions as above satisfying \([1.4] - [1.7] \). Let \( 0 < p < 1 \) and \( w \in A^p_i, i = 1, 2 \). the weighted Hardy spaces \( H^p_{(i),w}(\mathbb{R}^N) \), associated with \( \circ_i \) can be
defined by
\[ H^p_{\psi,w}(\mathbb{R}^N) = \{ f \in S'/P(\mathbb{R}^N) : \|g(i) f\|_{L^p_w(\mathbb{R}^N)} < \infty \}. \]
If \( \circ_1 \neq \circ_2 \), in general \( T_1 \) is bounded only on \( H^p_{(1),w} \) for the same \( i \), but not bounded on the other weighted Hardy space even if \( w \in A^{(1)}_\infty \cap A^{(2)}_\infty \). So, in general, the composition \( T_1 \circ T_2 \) is bounded neither on \( H^p_{(1),w} \) nor on \( H^p_{(2),w} \) when \( w \in A^{(1)}_\infty \cap A^{(2)}_\infty = A^c_\infty \).

(ii) If both \( T_1 \) and \( T_2 \) are smooth operators as in \( \text{PS} \) and \( \text{HLLRS} \) but satisfying the weaker cancellation conditions \( (1.3) \) (i.e. the size condition \( (1.1) \) and regularity condition \( (1.2) \) are replaced by appropriate differential inequalities), then the composition \( T_1 \circ T_2 \) is bounded on \( H^p_{\psi,w}(\mathbb{R}^N) \) for all \( 0 < p < \infty \), provided \( w \in A^c_\infty(\mathbb{R}^N) \). Thus the composition operators under consideration covers those operators studied in \( \text{PS} \) and \( \text{HLLRS} \) and our results are new even in the unweighted case, in view of the weaker cancellation condition \( (1.3) \).

We also prove the following general principle for a linear operator to be bounded from \( H^p_{\psi,w}(\mathbb{R}^N) \) to \( L^p_w(\mathbb{R}^N) \), which is of independent interest.

**Theorem 1.4.** Suppose \( w \in A^c_\infty(\mathbb{R}^N) \) and \( 0 < p \leq 1 \). Then any linear operator \( T \) which is bounded both on \( L^2(\mathbb{R}^N) \) and \( H^p_{\psi,w}(\mathbb{R}^N) \), is bounded from \( H^p_{\psi,w}(\mathbb{R}^N) \) to \( L^p_w(\mathbb{R}^N) \).

By Theorem \( 1.3 \) and Theorem \( 1.4 \) we have

**Theorem 1.5.** Suppose \( w \in A^c_\infty(\mathbb{R}^N) \). Then for \( q_w \{1 + \left( \frac{\varepsilon_1}{N_1} \wedge \frac{\varepsilon_2}{N_2} \right) \}^{-1} < p \leq 1 \), the composition operator \( T_1 \circ T_2 \) is bounded from \( H^p_{\psi,w}(\mathbb{R}^N) \) to \( L^p_w(\mathbb{R}^N) \). Moreover, there exists a constant \( C \) depending only on the dimensions and \( w \) such that
\[ \| (T_1 \circ T_2) f \|_{L^p_w(\mathbb{R}^N)} \leq C \| f \|_{H^p_{\psi,w}(\mathbb{R}^N)}. \]

Finally, we make the following remarks. Firstly, as mentioned above, the whole theory of weighted Hardy spaces are closely related to a family of rectangles of acceptable size, which reflect the geometry structure of \( \mathbb{R}^N \) with mixed homogeneities. Secondly, our methods to obtain the weighted norm inequalities in \( H^p_{\psi,w} \) are based on the weighted Littlewood-Paley-Stein theory and Fefferman-Stein vector-valued inequality, which are quite different from the classical method via atomic decompositions as in \( \text{LL} \) \( \text{LLY} \) \( \text{K} \). Thirdly, if \( \circ_1 = \circ_2 \), then the \( A^c_p \) weight is just one-parameter \( A_p \) weight and the weighted Hardy spaces \( H^p_{\psi,w}(\mathbb{R}^N) \) coincides with the non-isotropic weighted Hardy spaces \( H^p_w(\mathbb{R}^N) \) with equivalent norms.

Throughout this paper, let \( \mathbb{N} \) be the set of natural numbers \( \{0, 1, 2, \ldots \} \) and \( \mathbb{Z} \) be the set of integers. The letters \( C \) will always be used to denote positive constant only depending
on the dimensions and any other specified quantities. The values of the constant are not necessarily the same at each occurrence. We use $A \lesssim B$ to denote $A \leq CB$, and we write $A \approx B$ if $A \lesssim B \lesssim A$.

2. Proof of Theorem 1.1

We prove Theorem 1.1 by showing $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (1)$.

Proof of $(1) \Rightarrow (2)$. Note that $w \in A^G_{\infty}(\mathbb{R}^N)$ implies that $q_w < \infty$. It is worthwhile to note that the set $\mathcal{R}_G$ contains both families of the cubes with respect to the two dilations $\circ_1$ and $\circ_2$. Hence we have the inclusion relationship:

$$A^G_p(\mathbb{R}^N) \subset A_p(1) \cap A_p(2)(\mathbb{R}^N), \quad \text{for every } 1 < p < \infty,$$

which gives $(1) \Rightarrow (2)$.

Proof of $(2) \Rightarrow (3)$. Obviously.

Proof of $(3) \Rightarrow (4)$. This implication follows immediately from

Lemma 2.1. There is a constant $C$ so that $\mathcal{M}_G \lesssim \mathcal{M}_{(1)} \circ \mathcal{M}_{(2)}$, where each $\mathcal{M}_{(i)}$ is the Hardy-Littlewood maximal function associated with dilations $\circ_i$ on $\mathbb{R}^N$.

Proof. For any $j, k \in \mathbb{Z}$, by relabeling the coordinates if necessary, we can assume that $j_i = k_i$ for $i = 1, \ldots, l$, and $j_i \neq k_i$ for $i = l + 1, \ldots, m$. Then for every $x \in \mathbb{R}^N$, we write $x = (x^{(1)}, x^{(2)}) \in \mathbb{R}^l \oplus \mathbb{R}^{N-l}$. For any $G$-rectangle $R \in \mathcal{R}_G^{j,k}$ centered at origin, let $I_1$ be the “cube” in $\mathbb{R}^N$ centered at origin with sidelength $2^{j_1}, \ldots, 2^{j_l}$, and $I_2$ the cube centered at origin in $\mathbb{R}^{N-l}$ with side lengths $(2^{k_{l+1}}, \ldots, 2^{k_m})$. Let $\eta_R$, $\eta_{I_1}$, and $\eta_{I_2}$ be the normalized characteristic functions $\eta_R = |R|^{-1} \chi_R$, $\eta_{I_1} = |I_1|^{-1} \chi_{I_1}$, and $\eta_{I_2} = |I_2|^{-1} \chi_{I_2}$, respectively.

Write

$$
\chi_{I_1} \ast (\delta_{2^l} \otimes \chi_{I_2})(x) = \int_{I_2} \chi_{I_1}(x^{(1)}, x^{(2)} - y) \chi_{I_2}(y) dy.
$$

Note that for $x \in cR$ with $c$ small enough, we have $|x_i| \leq 2^{i \vee k_i - 2}$ for $i = 1, \ldots, m$. Thus for $i = l + 1, \ldots, m$ and $y \in I_2$, we have $|x_i - y_i| \leq |x_i| + |y_i| \leq 2^{i \vee k_i - 1} + 2^{k_i - 2} < 2^{i \vee k_i}$. Therefore the integrand is 1 whenever $x \in 1/4R$ and thus the integral exceeds $|I_2|$. Hence for $x \in R$,

$$
\eta_{I_1} \ast (\delta_{2^l} \otimes \eta_{I_2})(x) \geq \frac{|I_2|}{|I_1||R|} \geq \frac{1}{|R|} \gtrsim \eta_{cR}(x),
$$

which implies Lemma 2.1.\hfill \Box

Proof of $(4) \Rightarrow (1)$. The proof is rather standard as in the one-parameter case. We only sketch the proof for the reader’s convenience. We assume now $\mathcal{M}_G$ is bounded on $L^p_w$.
for a non-negative locally integrable function $w$. Apply this to the function $f_{\chi_R}$ supported in a $C$-rectangle $R$ and use that $1/|R| \int_R |f| \lesssim M\phi(f_{\chi_R})(x)$ for all $x \in R$ to obtain
\begin{equation}
(2.1) \quad w(R) \cdot \left( \frac{1}{|R|} \int_R |f(x)| \, dx \right)^p \leq C \int_R M\phi(f_{\chi_B})^p \, w(x) \, dx \leq C_p \int_R |f|^p \, w(x) \, dx,
\end{equation}
where $w(R) = \int_R w(x) \, dx$. It follows that
\begin{equation}
(2.2) \quad \left( \frac{1}{|R|} \int_R |f(t)| \, dt \right)^p \leq \frac{C_p}{w(R)} \int_R |f(x)|^p \, w(x) \, dx,
\end{equation}
for all balls $R \in \mathcal{R}_\phi$ and all functions $f$. Now we take $f = w^{-\nu'/p}$, which gives $f^p w = w^{-\nu'/p}$. We thus get that $w$ should satisfy the inequality (1) under additional assumption that $\inf_R w > 0$ for all $C$-rectangles $R$. If $\inf_R w = 0$ for some $C$-rectangles $R$, we take $f = (w + \varepsilon)^{-\nu'/p}$. Repeating the similar argument, we can derive
\begin{equation}
\left( \frac{1}{|R|} \int_R w(x) \, dx \right) \left( \frac{1}{|R|} \int_R (w(x) + \varepsilon)^{-\nu'/p} \, dx \right)^{p-1} \leq C_p,
\end{equation}
from which we can still get the conclusion (1) via the Lebesgue monotone convergence theorem by letting $\varepsilon \to 0$. This ends the proof of the implication (4) $\Rightarrow$ (1) and Theorem 1.1 follows. \hfill \Box

**Proof of Corollary 1.1.** The sufficiency part follows directly from Lemma 2.1, Theorem 1.1 and the one-parameter weighted Fefferman-Stein vector-valued inequality (see [AJ], [CG]). Concerning the necessity of $w \in A_C^\phi$, there is nothing to prove since $w \in A_C^\phi$ is already necessary in the scaler-valued case, $f = \{f_k\}$, $f_k = 0$, $k = 2, 3, \ldots$, according to Theorem 1.2. \hfill \square

3. The theory of weighted Hardy spaces

3.1. Some Lemmas. The following almost orthogonality estimate will be frequently used in the subsequent part of this section.

**Lemma 3.1.** (Almost orthogonality estimate) Given any positive integers $L$ and $M$, there exists a constant $C = C(L, M) > 0$ such that
\begin{equation}
|\psi_j,k \ast \varphi_{j',k'}(x)| \leq C 2^{-|j-j'|} L 2^{-|k-k'|} L \prod_{i=1}^m 2^{(j_i \vee j_i') (k_i \vee k_i')} M + |x_i| \right)^{n_i + M},
\end{equation}
where $\psi, \varphi$ are defined as in Section 1 (satisfying (1.4) - (3.3)).

**Remark 3.1.** The almost orthogonality estimate also holds if the functions $\phi, \psi$ only satisfy moment conditions up to order $M_0$
\begin{equation}
\int_{\mathbb{R}^N} \psi(x) x^\alpha \, dx = 0 = \int_{\mathbb{R}^N} \varphi(y) y^\beta \, dy \quad \text{for any multi-indices } |\alpha|, |\beta| \leq M_0.
\end{equation}
In this case, almost orthogonality estimate holds for all positive integers $M$ and all positive integers $L \leq M_0 + 1$.

The following useful estimate is also needed, see [HLLRS, Lemma 2.5].

**Lemma 3.2.** Let $R \in \mathcal{R}_\varphi^{j,k}$ Then for any $x \in R$, $x_R = (x_{R_1}, \ldots, x_{R_n}) \in R$, $x_{R'} = (x_{R_1'}, \ldots, x_{R_n'}) \in R'$ and for any $\frac{1}{1 + (\frac{M_1}{n_1} \wedge \ldots \wedge \frac{M_m}{n_m})} < \delta \leq 1$,

$$
\sum_{R' \in \mathcal{R}_{\varphi}^{j',k'}} |R'| \left[ \prod_{i=1}^{m} \frac{2^{M_i(j_i \vee j'_i \vee k_i \vee k'_i)}}{(2^{j_i \vee j'_i \vee k_i \vee k'_i} + |x_{R_i} - x_{R'_i}|)^{n_i + M_i}} \right] |g(x_{R'})| \lesssim C \left\{ \prod_{i=1}^{m} \left[ 2^{n_i(j_i - \ell_i)} \vee 1 \right] \right\} \frac{1}{2} \left\{ \mathcal{M}_{\varphi} \left[ \left( \sum_{R' \in \mathcal{R}_{\varphi}^{j',k'}} |g(x_{R'})|^2 \chi_{R'} \right)^{\frac{1}{2}} \right] (x) \right\} \frac{1}{2},
$$

where $C$ is a constant depending only on $M$, and the dimensions $n_1, \ldots, n_m$.

Using a idea similar as in the Shannon sampling theorem, we can prove the following discrete Calderón’s reproducing formula associated with different homogeneities. The proof is essentially similar [HLLRS, Proof of Theorem 1.3] and thus will be omitted. For the classical case, see [FJ, FJW].

**Theorem 3.1.** Suppose $\psi_{j,k}$ is defined as in Section 1 (satisfying (1.4)–(3.3)). Then

$$
(3.2) \quad f(x) = \sum_{j,k \in \mathbb{Z}} \sum_{R \in \mathcal{R}_{\varphi}^{j,k}} |R| \psi_{j,k} \ast f(x_R) \psi_{j,k}(x - x_R),
$$

where $x_R = (2^{j_1 \vee k_1} \ell_1, \ldots, 2^{j_m \vee k_m} \ell_m)$ is the minimal corner of $R$ and the series converges in $L^2(\mathbb{R}^N)$, $S_\infty(\mathbb{R}^N)$ and $S'/P(\mathbb{R}^N)$.

**3.2. Proof of Theorem 1.2.** Let $f \in S'/P(\mathbb{R}^N)$ and let $w \in A^p_\infty(\mathbb{R}^N)$. Let $x_R$ and $x_{R'}$ denote the minimal corner of $R$ and $R'$, respectively. Applying the discrete Calderón’s reproducing formula in Theorem 3.1, the almost orthogonality estimates with $M > N[(\frac{m}{p} - 1) \vee 0]$ and $L = 10M$, and Lemma 3.2 with $M_i = M$, we deduce that for
Lemma 3.2. and finally applying Hölder’s inequality, we obtain that for all 

\[
\frac{N}{N+M} < \delta < \left(\frac{p}{q_w} \wedge 1\right), \\
\|\psi_{j,k} \ast f\|_{L^R}(x_R)
\]

\[
\sim \sum_{j',k' \in \mathbb{Z}} \sum_{R' \in \mathcal{R}_{\ell}^{j',k'}} |R'| \varphi_{j,k} \ast f(x_R) |\psi_{j,k} \ast \varphi_{j',k'}(x_R - x_{R'})| \leq \sum_{j',k' \in \mathbb{Z}} 2^{-\|j-j'\|L_2 - \|k-k'\|L}
\]

\[
\times \sum_{R' \in \mathcal{R}_{\ell}^{j',k'}} |R'| \left[ \prod_{i=1}^{m} \left( \frac{2M(j_i \vee j_i' \vee k_i \vee k_i')}{(2i \vee j_i' \vee k_i' \vee k_i')} + \frac{|x_{R_i} - x_{R_i'}|}{\delta} \right)^{\eta_i + M} \right] \varphi_{j',k'} \ast f(x_{R'})
\]

\[
\lesssim \sum_{j',k' \in \mathbb{Z}} 2^{-\|j-j\|L' - \|k-k'\|L'} \left\{ \mathcal{M}_{\ell'} \left( \sum_{R' \in \mathcal{R}_{\ell'}^{j',k'}} |\varphi_{j',k'} \ast f(x_{R'})|^2 \chi_{R'} \frac{\delta}{2} \right) (x) \right\} \frac{1}{2},
\]

where \(\|\mathbf{i}\| = |j_1| + \ldots + |j_m|\), \(L' = L - N(1/\delta - 1)\) and the last inequality follows from Lemma 3.2.

Squaring both sides, then multiplying \(\chi_{R_i}\), summing over all \(j, k \in \mathbb{Z}\) and \(R \in \mathcal{R}_{\ell}^{j,k}\), and finally applying Hölder’s inequality, we obtain that for all \(x \in \mathbb{R}^N\), and \(\frac{N}{N+M} < \delta < \left(\frac{p}{q_w} \wedge 1\right), \)

\[
|g_{\ell}^\psi(f)(x)|^2 \lesssim \sum_{j,k \in \mathbb{Z}} \left\{ \sum_{j',k' \in \mathbb{Z}} 2^{-\|j-j\|L' - \|k-k'\|L'} \left\{ \sum_{j',k' \in \mathbb{Z}} 2^{-\|j-j\|L' - \|k-k'\|L'} \times \left\{ \mathcal{M}_{\ell'} \left( \sum_{R' \in \mathcal{R}_{\ell'}^{j',k'}} |\varphi_{j',k'} \ast f(x_{R'})|^2 \chi_{R'} \frac{\delta}{2} \right) (x) \right\} \right\} \frac{1}{2}
\]

\[
\lesssim \sum_{j',k' \in \mathbb{Z}} \left\{ \mathcal{M}_{\ell'} \left( \sum_{R' \in \mathcal{R}_{\ell'}^{j',k'}} |\varphi_{j',k'} \ast f(x_{R'})|^2 \chi_{R'} \frac{\delta}{2} \right) (x) \right\} \frac{1}{2},
\]

where in the last inequality we have used the inequalities 

\[
\sum_{j',k' \in \mathbb{Z}} 2^{-\|j-j\|L' - \|k-k'\|L'} \leq C \quad \text{and} \quad \sum_{j,k \in \mathbb{Z}} 2^{-\|j-j\|L' - \|k-k'\|L'} \leq C.
\]

Hence by the weighted Fefferman-Stein’s vector-valued inequality in Corollary 1.1 on \(L^p_{\ell}(|\mathbb{L}|^2/\delta)\) (noting that \((2 \wedge p)/\delta > q_w\) implies \(w \in A^p_{\ell}\delta(\mathbb{R}^N)\)), we obtain 

\[
\|g_{\ell}^\psi(f)\|_{L^p_{\ell}(\mathbb{R}^N)} \lesssim \|g_{\ell}^\psi(f)\|_{L^p_{\ell}(\mathbb{R}^N)}.
\]

By symmetry, we get the converse inequality. This concludes the proof of Theorem 1.2. □
From Theorem 1.2 we know that the definition of weighted Hardy spaces associated with different homogeneities is independent of particular choice of \( \psi_{j,k} \) and thus is well defined.

**Corollary 3.1.** Let \( w \in A^\infty_\infty(\mathbb{R}^N) \). Then \( S_\infty(\mathbb{R}^N) \) is dense in \( H_{\varepsilon,w}^p(\mathbb{R}^N) \) for \( 0 < p < \infty \).

**Proof.** Let \( f \in H_{\varepsilon,w}^p(\mathbb{R}^N) \) and let \( \psi_{j,k} \) be the same as in Theorem 1.2. For any fixed \( n > 0 \), denote

\[
E_n = \{(j, k, R) : \forall_{i=1}^m |j_i \lor k_i| \leq n, \ R \in \mathcal{R}_{\varepsilon,q}^{j,k}, \ R \subset B(0, n)\},
\]

and

\[
f_n(x) = \sum_{(j, k, R) \in E_n} 2^{j_1 \lor k_1 + \cdots + j_m \lor k_m} \psi_{j,k}(x) \psi_{j,k}(x - x_R).
\]

Since \( \psi_{j,k} \in S_\infty(\mathbb{R}^N) \), we obviously have \( f_n \in S_\infty(\mathbb{R}^N) \). Repeating the proof of Theorem 1.2, we conclude that \( \|f_n\|_{H_{\varepsilon,w}^p(\mathbb{R}^N)} \leq C \|f\|_{H_{\varepsilon,w}^p(\mathbb{R}^N)} \). To see that \( f_n \) tends to \( f \) in \( H_{\varepsilon,w}^p(\mathbb{R}^N) \), by the discrete Calderón’s reproducing formula,

\[
(f - f_n)(x) = \sum_{(j, k, R) \in \mathcal{E}_{E_n}} 2^{j_1 \lor k_1 + \cdots + j_m \lor k_m} \psi_{j,k}(x) \psi_{j,k}(x - x_R),
\]

where the series converges in \( S_\infty(\mathbb{R}^N) \) and \( S'/P(\mathbb{R}^N) \). Therefore, \( g_\varepsilon(f - f_n)(x)^2 \) equals

\[
\sum_{j', k' \in \mathbb{Z}} \sum_{R' \in \mathcal{R}_{\varepsilon,q}^{j', k'}} \sum_{(j, k, R) \in \mathcal{E}_{E_n}} 2^{j_1 \lor k_1 + \cdots + j_m \lor k_m} \psi_{j,k}(x) \psi_{j,k}(x - x_R)^2 \chi_{R'}(x).
\]

Now repeating the proof of Theorem 1.2 we get that

\[
\|g_\varepsilon(f - f_n)\|_{L^p_w(\mathbb{R}^N)} \leq C \left\| \left\{ \sum_{(j, k, R) \in \mathcal{E}_{E_n}} |\psi_{j,k}(x) \psi_{j,k}(x - x_R)|^2 \chi_R \right\}^{\frac{1}{2}} \right\|_{L^p_w(\mathbb{R}^N)},
\]

where the last term tends to 0 as \( n \) goes to infinity. This implies that \( f_n \) tends to \( f \) in \( H_{\varepsilon,w}^p(\mathbb{R}^N) \) norm and hence the proof of Corollary 3.1 is finished. \( \Box \)

### 3.3. Calderón’s reproducing formula involving bump functions with compact support

We also need a new Calderón’s reproducing formula involving bump function with compact support. Let \( \phi^{(1)}(x) \in \mathcal{S}(\mathbb{R}^N) \) supported in the unit sphere in \( \mathbb{R}^N \) with

\[
\int_{\mathbb{R}^N} \phi^{(1)}(x)x^{\alpha_1}dx = 0, \text{ for } 0 \leq |\alpha_1| \leq M_0,
\]

where \( M_0 \) is a large positive integer which will be determined later, and

\[
\sum_{j \in \mathbb{Z}} \hat{\phi}^{(1)}(2^j \circ \xi) = 1, \text{ for all } \xi \in \mathbb{R}^N \setminus \{0\}.
\]
with \( \phi^{(2)} \) satisfying similar conditions with respect to \( \circ_2 \). For \( j, k \in \mathbb{Z} \), let \( \phi^{(1)}_j(x) = 2^{-jN_1}\phi^{(1)}(2^{-j} \circ_1 x) \), \( \phi^{(2)}_k(x) = 2^{-kN_2}\phi^{(2)}(2^{-k} \circ_2 x) \) and \( \phi_{j,k}(x) = \phi^{(1)}_j * \phi^{(2)}_k(x) \).

**Theorem 3.2.** Let \( 0 < p \leq 1 \) and \( w \in A^p_{\infty}(\mathbb{R}^N) \). Let \( \phi_{j,k} \) be defined as above with \( M_0 \geq 10(N[q_w/(1 \wedge p) - 1] + 1) \). Here \( [\cdot] \) denotes the greatest integer function. Then for any \( f \in L^2(\mathbb{R}^N) \cap H^p_{\epsilon,w}(\mathbb{R}^N) \), there exists \( h \in L^2(\mathbb{R}^N) \cap H^p_{\epsilon,w}(\mathbb{R}^N) \) such that for a sufficiently large \( K \in \mathbb{N} \),

\[
\tag{3.4} f(x) = \sum_{j,k \in \mathbb{Z}} \sum_{R \in \mathcal{R}_{\epsilon}^{j-K,k-K}} |R| \phi_{j,k}(x - x_R) \phi_{j,k} * h(x_R),
\]

where \( x_R \) denotes the minimal corner of dyadic \( \epsilon \)-rectangle \( R \) and the series converges in \( L^2(\mathbb{R}^N) \). Moreover,

\[
\tag{3.5} \|f\|_{L^2(\mathbb{R}^N)} \sim \|h\|_{L^2(\mathbb{R}^N)}.
\]

and

\[
\tag{3.6} \|f\|_{H^p_{\epsilon,w}(\mathbb{R}^N)} \sim \|h\|_{H^p_{\epsilon,w}(\mathbb{R}^N)}.
\]

**Proof.** Applying Coifman’s decomposition of the identity operator, we have

\[
\begin{align*}
\sum_{j,k \in \mathbb{Z}} \sum_{R \in \mathcal{R}_{\epsilon}^{j-K,k-K}} |R| \phi_{j,k} * f(x_R) \phi_{j,k}(x - x_R) + S_K(f)(x) := T_K(f)(x) + S_K(f)(x),
\end{align*}
\]

where

\[
\begin{align*}
S_K(f)(x) &= \sum_{j,k \in \mathbb{Z}} \sum_{R \in \mathcal{R}_{\epsilon}^{j-K,k-K}} \int_R \phi_{j,k}(x - x') (\phi_{j,k} * f)(x') - \phi_{j,k}(x - x_R) (\phi_{j,k} * f)(x_R) dx'
&= \sum_{j,k \in \mathbb{Z}} \sum_{R \in \mathcal{R}_{\epsilon}^{j-K,k-K}} \int_R \left[ \phi_{j,k}(x - x') - \phi_{j,k}(x - x_R) \right] (\phi_{j,k} * f)(x') dx'
&\quad + \sum_{j,k \in \mathbb{Z}} \sum_{R \in \mathcal{R}_{\epsilon}^{j-K,k-K}} \int_R \phi_{j,k}(x - x') [(\phi_{j,k} * f)(x') - (\phi_{j,k} * f)(x_R)] dx'
&:= S^1_K(f)(x) + S^2_K(f)(x),
\end{align*}
\]

Now we claim that for \( k = 1, 2 \),

\[
\tag{3.7} \|R^k_K(f)\|_{L^2(\mathbb{R}^N)} \leq C 2^{-K} \|f\|_{L^2(\mathbb{R}^N)},
\]

and

\[
\tag{3.8} \|R^k_K(f)\|_{H^p_{\epsilon,w}(\mathbb{R}^N)} \leq C 2^{-K} \|f\|_{H^p_{\epsilon,w}(\mathbb{R}^N)},
\]

where \( C \) is a constant independent of \( f \) and \( K \).
Assume the claim for the moment, then, by choosing sufficiently large \( K \), \( T_K^{-1} = \sum_{n=0}^{\infty} (S_K)^n \) is bounded on both \( L^2(\mathbb{R}^N) \) and \( H^p_{\ell, w}(\mathbb{R}^N) \). For any \( f \in L^2(\mathbb{R}^N) \cap H^p_{\ell, w}(\mathbb{R}^N) \), set \( h = T_K^{-1}(f) \), then the estimates in (3.7) and (3.8) imply (3.5) and (3.6). Moreover,

\[
f(x) = T_K(T_K^{-1}(f))(x) = \sum_{j,k \in \mathbb{Z}} \sum_{R \in \mathcal{R}_{\ell}^{j-k-k}} |R| \phi_{j,k}(x-x_R)(\phi_{j,k} * h)(x_R),
\]

where the series converges in \( L^2(\mathbb{R}^N) \).

Thus, to finish the proof of Theorem 3.2, it suffices to verify the claim. Since the proofs for \( S^1_\ell \) and \( S^2_\ell \) are similar, we only give the proof for \( S^1_\ell \). Roughly speaking, the proof is similar to Theorem 1.2. To see this, let \( f \in L^2(\mathbb{R}^N) \cap H^p_{\ell, w}(\mathbb{R}^N) \), applying discrete Calderón’s reproducing formula in Theorem 3.1 yields

\[
\psi_{j', k'} * S^1_\ell(f)(x) = \sum_{j,k \in \mathbb{Z}} \sum_{R \in \mathcal{R}_{\ell}^{j-k-k}} \int_{R} \psi_{j', k'} * [\phi_{j,k}(\cdot - x') - \phi_{j,k}(\cdot - x_R)](x)(\phi_{j,k} * f)(x')dx' = \sum_{j,k \in \mathbb{Z}} \sum_{R \in \mathcal{R}_{\ell}^{j-k-k}} \int_{R} \psi_{j', k'} * [\phi_{j,k}(\cdot - x') - \phi_{j,k}(\cdot - x_R)](x) \times \left( \sum_{j'' \in \mathbb{Z}} \sum_{R'' \in \mathcal{R}_{\ell}^{j''-k-k}} |R''| \cdot \psi_{j'', k''} * f(x_{R''}) \phi_{j'', k''}(x' - x_{R''}) \right)dx',
\]

where \( x''_R = (x''_{R,1}, \ldots, x''_{R,n}) \) is the minimal corner of \( R'' \).

Set \( \tilde{\phi}_{j,k}(u) = \phi_{j,k}(u-x') - \phi_{j,k}(u-x_R) \). Then applying Lemma 3.1 (particularly Remark 3.1) with \( M = N[q_w/(1 + p) - 1] + 1 \) and \( L = 10M \), we obtain that for some constant \( C \) (depending only on \( M, \psi \) and \( \phi \), but independent of \( K \)),

\[
|\psi_{j', k'} * \tilde{\phi}_{j,k}(x)| \leq C2^{-K}2^{-10M||j-j'||2-10M||k-k'||} \prod_{i=1}^{m} \frac{2^{(j_i \wedge j'_i) \vee k_i \wedge k'_i)M}}{(2^{j_i \wedge k_i \vee k'_i} + |x_i - x'_i|)^{n_i + M}} \leq C2^{-K}2^{-3M||j-j'||2-3M||k-k'||} \prod_{i=1}^{m} \frac{2^{j_i \wedge k_i \vee k'_i)M}}{(2^{j_i \wedge k_i \vee k'_i} + |x_i - x'_i|)^{n_i + M}},
\]

where the last inequality follows from \( 2^{j_i \wedge j'_i \vee k_i \vee k'_i} \leq 2^{||j-j'||2||k-k'||2^{j_i \wedge k_i \vee k'_i}. \) Similarly

\[
|\phi_{j,k} * \psi_{j'', k''}(x' - x_{R''})| \leq C2^{-K}2^{-3M||j-j''||2-3M||k-k''||} \prod_{i=1}^{m} \frac{2^{(j''_i \wedge k'_i)M}}{(2^{j''_i \wedge k'_i} + |x'_i - x''_{R,i}|)^{n_i + M}}.
\]
Inserting these estimates into the last term in (3.9) yields
\[
\big|\psi_{j',k'} \ast S^1_R(f)(x)\big| \\
\lesssim \sum_{j'',k'' \in \mathbb{Z}} \sum_{R'' \in \mathcal{R}_d} |R''| |\psi_{j'',k''} \ast f(x_{R''})| \\
\times \sum_{j,k \in \mathbb{Z}} \sum_{R \in \mathcal{R}_d} \int_R 2^{-K} \prod_{i=1}^m 2^{-|I_j|} |H_k| |S_M| |R''| |K''|^{M} \\
\times 2^{-|I_j''|} |H_k''| |R''| |K''|^{M} \prod_{i=1}^m (2(|I_j''| + |x_i - x_{R''}|))^{n_i + M} dx' \\
\lesssim 2^{-K} \sum_{j'',k'' \in \mathbb{Z}} \sum_{R'' \in \mathcal{R}_d} 2^{-|I_j''|} |H_k''| |R''| |K''|^{M} |R''| \\
\times \left\{ \prod_{i=1}^m \frac{2(|I_j''| + |x_i - x_{R''}|))^{n_i + M}}{(2|M_k''| + |x_i - x_{R''}|))^{n_i + M}} \right\} \big|\psi_{j'',k''} \ast f(x_{R''})\big|.
\]

Since \( M = N[q_w/(1 \land p) - 1] + 1 \), we can choose \( N/(N + M) < \epsilon < 1 \) so that \( p/\epsilon > q_w \) and thus \( w \in A^\epsilon_{p/\epsilon}(\mathbb{R}^N) \). Applying Lemma 3.2 with \( M_i = M \) and the \( L^p_{p/\epsilon}(L^2/\epsilon) \) boundedness of \( \mathcal{M}_d \), we have
\[
\|S^1_R(f)\|_{H^p_{p/\epsilon}\mathbb{R}^N} \lesssim \|g\psi[S^1_R(f)]\|_{L^p_{p/\epsilon}\mathbb{R}^N} \\
\lesssim 2^{-K} \left\{ \sum_{j'',k'' \in \mathbb{Z}} \left\{ \mathcal{M}_d \left( \sum_{R'' \in \mathcal{R}_d} |\psi_{j'',k''} \ast f(x_{R''})| \right) \right\} \right\}^{1/2} \\
\lesssim 2^{-K} \left\{ \sum_{j'',k'' \in \mathbb{Z}} \sum_{R'' \in \mathcal{R}_d} |\psi_{j'',k''} \ast f(x_{R''})|^2 \right\}^{1/2} \lesssim 2^{-K} \|f\|_{H^p_{p/\epsilon}\mathbb{R}^N}.
\]
The claim is concluded and hence Theorem 3.2 follows. \( \square \)

Using the similar argument as in the proof of Theorem 3.2, we can get

**Corollary 3.2.** Suppose \( w \in A^\epsilon_{\infty}(\mathbb{R}^N) \). Then for \( f \in L^2(\mathbb{R}^N) \cap H^p_{p/\epsilon\mathbb{R}^N} \) and \( 0 < p \leq 1 \), there exists a function \( h \) such that for sufficient large \( K \),
\[
\|f\|_{H^p_{p/\epsilon\mathbb{R}^N}} \approx \|g\psi(f)\|_{L^p_{p/\epsilon\mathbb{R}^N}} \equiv \left\{ \sum_{j,k \in \mathbb{Z}} \sum_{R \in \mathcal{R}_d} |\phi_{j,k} \ast h(x_R)|^2 \chi_R(\cdot) \right\}^{1/2}.
\]

4. \( H^p_{p/\epsilon\mathbb{R}^N} \) weighted norm inequalities for composition operators

4.1. Proof of Theorem 1.3.
Lemma 4.1. Let $\phi_j^{(1)}$, $\phi_{j'}^{(1)}$, $\phi_k^{(2)}$, $\phi_{k'}^{(2)}$ be the functions as in the last section with $M_0 \geq 10\varepsilon$. Then

\begin{equation}
|\phi_j^{(1)} \ast K_1 \ast \phi_{j'}^{(1)}(x)| \leq C 2^{-10\varepsilon \|j-j'\|} \prod_{i=1}^m \frac{2^{(i,\nu j')} \varepsilon_i}{(2^{i,\nu j'} + |x_i|)^{n_i + \varepsilon_i}}.
\end{equation}

and

\begin{equation}
|\phi_k^{(2)} \ast K_2 \ast \phi_{k'}^{(2)}(x)| \leq C 2^{-10\varepsilon \|k-k'\|} \prod_{i=1}^m \frac{2^{(k,\nu k')} \varepsilon_i}{(2^{k,\nu k'} + |x_i|)^{n_i + \varepsilon_i}},
\end{equation}

where $\varepsilon_k = \varepsilon_k n_i/N_k$ for $i = 1, \ldots, m$ and $k = 1, 2$.

Proof of Lemma 4.1. We only show (4.1) as (4.2) can be proved similarly. By classical almost orthogonality estimate,

$$
\phi_j^{(1)} \ast \phi_{j'}^{(1)}(u) = C 2^{-10\varepsilon \|j-j'\|} \varphi_{j\vee j'}(u),
$$

with $\varphi_{j\vee j'}(u) = 2^{-(j+j')}N \varphi(2^{-j+j'} \circ_1 u)$ and $\varphi \in S(\mathbb{R}^N)$ supported in $\{|u| \leq 2\}$ with the same moment conditions as $\phi^{(1)}$. If we can show

\begin{equation}
|K_1 \ast \varphi(x)| \lesssim \frac{1}{(1 + |x_1|)^{N_1 + \varepsilon_1}},
\end{equation}

then a dilation argument would yield

$$
|\phi_j^{(1)} \ast K_1 \ast \phi_{j'}^{(1)}(x)| \lesssim 2^{-10\varepsilon \|j-j'\|} 2^{-(j+j')}N \frac{1}{(1 + |2^{-j+j'} \circ_1 x_1|)^{N_1 + \varepsilon_1}} \prod_{i=1}^m \frac{2^{(i,\nu j')} \varepsilon_i}{(2^{i,\nu j'} + |x_i|)^{n_i + \varepsilon_i}},
$$

which gives (4.1). Thus to finish the proof, it remains to establish (4.3).

If $|x_1| \geq 4$, then applying the cancellation condition of $\varphi$ and regularity condition of $K_1$ (since $|x_1| \geq 2|x_1|$),

$$
|K_1 \ast \varphi(x)| = \left| \int [K_1(x-u) - K_1(x)] \varphi(u) du \right| \lesssim \int \frac{|u|^{\varepsilon_1}}{|x_1|^{N_1 + \varepsilon_1}} |\varphi(u)| du \lesssim \frac{1}{(1 + |x_1|)^{N_1 + \varepsilon_1}}.
$$

If $|x_1| \leq 4$, then write

$$
|K_1 \ast \varphi(x)| = \left| \int_{|u| \leq 6} K_1(u)[\varphi(x-u) - \varphi(x)] du \right| + \left| \varphi(x) \right| \cdot \left| \int_{|u| \leq 6} K_1(u) du \right|.
$$

The first term can be estimated by the size condition of $K_1$ and smoothness condition of $\varphi$ while the second term can be dealt with by using the cancellation condition of $K_1$. This concludes the proof of (4.3) and Lemma 4.1 follows. \qed
We now turn to the

**Proof of Theorem 1.3.** We first assume that \( f \in L^2(\mathbb{R}^N) \cap H^p_{\varrho,w}(\mathbb{R}^N) \). By Corollary 3.2 and Theorem 3.2

\[
\|Tf\|_{H^p_{\varrho,w}(\mathbb{R}^N)} \lesssim \left\{ \sum_{j,k \in \mathbb{Z}} \sum_{R \in \mathbb{R}^j_{\varrho} - K_{j,K}} \left| \phi_{j,k} \ast K_1 \ast K_2 \ast f(x_R) \right|^2 \chi_R \right\}^{\frac{1}{2}} \left\| L^p_w(\mathbb{R}^N) \right\|
\]

\[
(4.4)
\]

where \( h = T_K^{-1}(f) \) and \( K \) are the same as in Theorem 3.2

From Lemma 4.1 it follows that

\[
|\phi_{j,k} \ast K_1 \ast K_2 \ast \phi_{j',k'}(x)| \leq C 2^{-10\varepsilon-j'\|2^{-10\varepsilon-k'\| \prod_{i=1}^m 2^{(i\vee j')\vee k'}}} \lesssim \frac{2^{(i\vee j')\vee k'} \varepsilon_i}{\delta + |x_i|^{n_i + \varepsilon_i}}.
\]

where \( \varepsilon_i = \varepsilon_i^1 \vee \varepsilon_i^2 \). Since \( p > q_w(1 + (\frac{N}{N_1} \vee \frac{N}{N_2}))^{-1} \), we can choose \( 1/(1 + \frac{N}{N_1} \vee \frac{N}{N_2}) < \delta < (1 + \frac{p}{q_w}) \) so that \( p/\delta > q_w \) and thus \( w \in \mathcal{A}^\varrho_{p/\delta}(\mathbb{R}^N) \). Substituting (4.5) into (4.4), applying Lemma 3.2 with \( M_i = \varepsilon_i^- \) (note that for \( i = 1, \ldots, m, \frac{\varepsilon_i^-}{n_i} = \frac{\varepsilon_i^1}{N_1} \wedge \frac{\varepsilon_i^2}{N_2} \)) and the Fefferman-Stein vector-valued inequality on the \( L^p_w(\ell^{2/\delta}) \) boundedness of \( \mathcal{M}_\varrho \), we deduce that for \( f \in L^2(\mathbb{R}^N) \cap H^p_{\varrho,w}(\mathbb{R}^N) \),

\[
\|Tf\|_{H^p_{\varrho,w}(\mathbb{R}^N)} \lesssim \left\{ \sum_{j',k' \in \mathbb{Z}} \mathcal{M}(\sum_{R' \in \mathbb{R}^j_{\varrho,K_{j,K}}} \left| \phi_{j',k'} \ast h(x_{R'}) \right|^2 \chi_{R'})^{\frac{1}{2}} \right\}^{\frac{1}{2}} \| L^p_w(\mathbb{R}^N) \|
\]

which, after a standard density argument (in view of Lemma 3.1), concludes the proof of Theorem 1.3.\( \square \)

4.2. **Proof of Theorem 1.4.** To prove Theorem 1.4 we need the following

**Theorem 4.1.** Suppose \( w \in A^\varrho_{\infty}(\mathbb{R}^N) \). If \( f \in L^2(\mathbb{R}^N) \cap H^p_{\varrho,w}(\mathbb{R}^N), 0 < p \leq 1 \), then \( f \in L^p_w(\mathbb{R}^N) \) and there is a constant \( C > 0 \) which is independent of the \( L^2(\mathbb{R}^N) \) norm of \( f \) such that

\[
\|f\|_{L^p_w(\mathbb{R}^N)} \leq C \|f\|_{H^p_{\varrho,w}(\mathbb{R}^N)}.
\]
Proof. Without loss the generality, we may assume that $w \in A^q_v(\mathbb{R}^N)$ for some $(2 \vee q)w \leq q < \infty$. Let $f \in L^2(\mathbb{R}^N) \cap H^p_{g,w}(\mathbb{R}^N)$, by Corollary 3.2 we have $\|f\|_{H^p_{g,w}(\mathbb{R}^N)} \approx \|	ilde{g}_e(f)\|_{L^p_w(\mathbb{R}^N)}$. Let $f \in L^2(\mathbb{R}^N) \cap H^p_{\varphi,w}(\mathbb{R}^N)$, by Corollary 3.2, we have $\|f\|_{H^p_{\varphi,w}(\mathbb{R}^N)} \approx \|	ilde{g}_e(f)\|_{L^p_w(\mathbb{R}^N)}$. For $i \in \mathbb{Z}$, set $\Omega_i = \{x \in \mathbb{R}^N : \tilde{g}_e(f)(x) > 2^i\}$. Denote

$$B_i = \{(j, k, R) : j, k \in \mathbb{Z}, R \in \mathcal{R}^{j, k, k}_\varepsilon, w(R \cap \Omega_i) > 1/2w(R), w(R \cap \Omega_{i+1}) \leq 1/2w(R)\}.$$ Then by the discrete Calderón reproducing formula in Theorem 3.2, we can write

$$f(x) = \sum_{i \in \mathbb{Z}} \sum_{(j, k, R) \in B_i} |R|\tilde{\varphi}_{j,k}(x - x_R)\varphi_{j,k} \ast h(x_R),$$

where the series converges in $L^2(\mathbb{R}^N)$ norm and hence $w$-almost everywhere. We claim that

$$\left(4.6\right) \left\| \sum_{i \in \mathbb{Z}} \sum_{(j, k, R) \in B_i} |R|\tilde{\varphi}_{j,k}(\cdot - x_R)\varphi_{j,k} \ast h(x_R) \right\|_{L^p_w(\mathbb{R}^N)}^p \lesssim 2^{pi}w(\Omega_i).$$

This claim together with the fact that $0 < p \leq 1$ would yield

$$\|f\|_{L^p_w(\mathbb{R}^N)}^p \leq \sum_{i \in \mathbb{Z}} \left(\sum_{(j, k, R) \in B_i} |R|\tilde{\varphi}_{j,k}(\cdot - x_R)\varphi_{j,k} \ast h(x_R) \right)_{L^p_w(\mathbb{R}^N)}^p \lesssim \sum_{i \in \mathbb{Z}} 2^{pi}w(\Omega_i) \lesssim \|	ilde{g}_e(f)\|_{L^p_w(\mathbb{R}^N)}^p \lesssim \|h\|_{H^p_{\varphi,w}(\mathbb{R}^N)}^p \lesssim \|f\|_{H^p_{\varphi,w}(\mathbb{R}^N)}^p,$$

which would imply Theorem 1.4.

Thus to finish the proof, it remains to show claim (4.6). Note that if $(j, k, R) \in B_i$, then $\tilde{\varphi}_{j,k}(x - x_R)$ is supported in $\tilde{\Omega}_i := \{x : \mathcal{M}_\varepsilon(\chi_{\Omega_i})(x) > 1/100\}$. Thus, by Hölder’s inequality,

$$\left(4.7\right) \left\| \sum_{(j, k, R) \in B_i} |R|\tilde{\varphi}_{j,k}(\cdot - x_R)\varphi_{j,k} \ast h(x_R) \right\|_{L^p_w(\mathbb{R}^N)}^p \leq Cw(\tilde{\Omega}_i)^{1-(p/q)} \sum_{(j, k, R) \in B_i} |R|\tilde{\varphi}_{j,k}(\cdot - x_R)\varphi_{j,k} \ast h(x_R)_{L^p_w(\mathbb{R}^N)}^p.$$


By the duality argument, for all \( \zeta \in L^{q'}_{w^{1-q'}}(\mathbb{R}^N) \) with \( \| \zeta \|_{L^{q'}_{w^{1-q'}}(\mathbb{R}^N)} \leq 1 \), we have

\[
\left| \sum_{(j,k,R) \in B_i} |R| \bar{\phi}_{j,k}(-x) \phi_{j,k} * h(x_R), \zeta \right| \\
= \left| \sum_{(j,k,R) \in B_i} \int \bar{\phi}_{j,k} * \zeta(x_R) \phi_{j,k} * h(x_R) \chi_R(x) dx \right| \\
\leq \left\| \left\{ \sum_{(j,k,R) \in B_i} |\phi_{j,k} * h(x_R)|^2 \chi_R(\cdot) \right\}^{\frac{1}{2}} \right\|_{L^q_w(\mathbb{R}^N)} \\
\times \left\| \left\{ \sum_{(j,k,R) \in B_i} |\bar{\phi}_{j,k} * \zeta(x_R)|^2 \chi_R(\cdot) \right\}^{\frac{1}{2}} \right\|_{L^{q'}_{w^{1-q'}}(\mathbb{R}^N)} \\
\equiv J_1 \times J_2,
\]

where \( \bar{\phi}_{j,k}(x) = \bar{\phi}_{j,k}(-x) \).

We first estimate \( J_1 \). Note that \( \Omega_i \subset \tilde{\Omega}_i \), and by the \( L^q_w(\mathbb{R}^N) \) boundedness of \( \mathcal{M}_\varepsilon \), \( w(\tilde{\Omega}_i) \leq Cw(\Omega_i) \). For any \( (j,k,R) \in B_i \), if \( x \in R \) then \( \mathcal{M}_\varepsilon(\chi_{R \cap (\tilde{\Omega}_i \setminus \Omega_{i+1})})(x) > \frac{1}{2} \) and therefore \( \chi_R(x) \leq 2\mathcal{M}_\varepsilon(\chi_{R \cap (\tilde{\Omega}_i \setminus \Omega_{i+1})})(x) \). Thus, using the weighted Fefferman-Stein inequality in Corollary 1.1 again, we have

\[
J_1^q = \| \left\{ \sum_{(j,k,R) \in B_i} |\phi_{j,k} * h(x_R)|^2 \chi_R(\cdot) \right\}^{\frac{1}{2}} \|_{L^q_w(\mathbb{R}^N)}^q \\
= \int_{\mathbb{R}^N} \left\{ \sum_{(j,k,R) \in B_i} |\phi_{j,k} * h(x_R)|^2 \chi_R(x) \right\}^{\frac{q}{2}} w(x) dx \\
\leq C \int_{\mathbb{R}^N} \left\{ \sum_{(j,k,R) \in B_i} |\phi_{j,k} * h(x_R)|^2 \mathcal{M}_\varepsilon(\chi_{R \cap (\tilde{\Omega}_i \setminus \Omega_{i+1})})(x) \right\}^{\frac{q}{2}} w(x) dx \\
\leq C \int_{\tilde{\Omega}_i \setminus \Omega_{i+1}} \left\{ \sum_{(j,k,R) \in B_i} |\phi_{j,k} * h(x_R)|^2 \chi_R(x) \right\}^{\frac{q}{2}} w(x) dx \\
\leq C 2^q w(\tilde{\Omega}_i) \leq C 2^q w(\Omega_i).
\]

As for \( J_2 \), since \( w \in A^q_{\varepsilon}(\mathbb{R}^N) \) implies \( w^{1-q'} \in A^q_{\varepsilon}(\mathbb{R}^N) \), by weighted Fefferman-Stein inequality in Corollary 1.1 we have

\[
J_2 \lesssim \| \tilde{g}_\varepsilon(\zeta) \|_{L^{q'}_{w^{1-q'}}(\mathbb{R}^N)} \lesssim 1.
\]
Combining the two estimates in (4.9) and (4.8), we obtain
\[
\left\| \sum_{(j,k,R) \in B_i} |R| \tilde{\phi}_{j,k}(\cdot - x_R)\phi_{j,k} * h(x_R) \right\|_{L^q(w_R(\mathbb{R}^N))} \lesssim 2^{iq}w(\Omega_i).
\]
Substituting this estimate into (4.7) yields claim (4.6) and hence Theorem 4.1 follows. □

Finally, we apply Theorem 4.1 to give the

**Proof of Theorem 1.4** For \( f \in L^2(\mathbb{R}^N) \cap H_{\varepsilon,w}^p(\mathbb{R}^N) \), by Theorems 4.1 and 1.3
\[
\|T(f)\|_{L^q(w_R(\mathbb{R}^N))} \leq C\|T(f)\|_{H_{\varepsilon,w}^p(\mathbb{R}^N)} \leq C\|f\|_{H_{\varepsilon,w}^p(\mathbb{R}^N)}.
\]
According to Corollary 3.1 a density argument yields Theorem 1.4. □

**References**


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